$N=2$ supersymmetry and dipole moments

Ioannis Giannakis* and James T. Liu†
Physics Department, The Rockefeller University, 1230 York Avenue, New York, New York 10021-6399
(Received 29 January 1998; published 18 June 1998)

We derive sum rules for the magnetic and electric dipole moments of all particle states of an $N=2$ supermultiplet. For short representations, we find agreement with previously determined $N=1$ sum rules, while there is added freedom for long representations (expressed as certain scalar expectation values). With mild assumptions we find the simple result that the supersymmetry generated spin adds to the magnetic (electric) dipole moment with a strength corresponding to $g=2$ $(g_e=0)$. This result is equally valid for $N=1$, this time without any further assumptions. [S0556-2821(98)03714-X]

PACS number(s): 12.60.Jv, 13.40.Em

I. INTRODUCTION

One of the great successes of the Dirac theory was its correct prediction of the gyromagnetic ratio of the electron. This was particularly striking since $g=2$ is twice as large as expected for the classical orbital motion of a charged point particle with angular momentum $\hbar/2$. However, there is nothing in Dirac's theory that requires a g value of 2 for a spin $J=1/2$ particle. Lorentz and gauge invariance do not prohibit the inclusion of a Pauli term into the Dirac equation. This term would provide an additional contribution to the magnetic moment of the electron and alter the value of g. A justification for its absence is that such a term would render the theory nonrenormalizable. Renormalizability is a matter of asymptotic behavior at infinite momentum, so it was not surprising that Weinberg [1] by demanding good asymptotic behavior for the photon forward-scattering amplitudes showed that $g=2$ for arbitrary spin charged particles that do not participate in the strong interactions. More recently, Ferrara, Porrati, and Telegdi [2] implemented this particular electromagnetic coupling prescription at the Lagrangian level. This prescription is different than the minimal coupling prescription according to which all derivatives ∂_μ are replaced by covariant ones D_μ and which yields $g=1/J$ for the gyromagnetic ratio of a particle of spin J [3].

The addition of supersymmetry leads to further consequences for the magnetic dipole moments. In Ref. [4] Ferrara and Remiddi showed that $g=2$ to all orders in perturbation theory for an $N=1$ chiral multiplet (superspin 0). On the other hand, when spin-1 fields (superspin 1/2) are involved, Bilchak, Gastmans, and Van Proeyen [5] showed that supersymmetry does not necessarily demand $g=2$, but nevertheless leads to a relation between the g-factors of the spin-1/2 and spin-1 particles of the superspin 1/2 multiplet. Subsequently, Ferrara and Porrati [6], utilizing only the supersymmetry algebra, found exact model independent sum rules relating the gyromagnetic ratios of all particles within a single massive $N=1$ supermultiplet. In particular, for a superspin j multiplet (with particles of spins $j-1/2$, j, j, and $j+1/2$), the gyromagnetic ratios may be expressed in terms of a single free parameter, namely the transition moment between the spin $j-1/2$ and $j+1/2$ states of the multiplet. Furthermore, when this nondiagonal moment vanishes, the sum rule simply states that $g=2$ for all members of the supermultiplet. In particular, this confirms the result of [4] that $g=2$ for a chiral multiplet since there is no room for a transition moment for superspin 0. Thus the anomalous magnetic moment of the electron (in a supersymmetric standard model) identically vanishes, as long as supersymmetry remains unbroken. This is but one of the manifestations of how supersymmetry alone provides powerful results independent of any particular model.

More recently, dipole moment sum rules have been applied in order to test the conjectured equivalence of string states and black holes. For this conjecture to be true, not only do masses, charges and representations have to agree, but so do other physical properties such as electric and magnetic dipole moments. Furthermore, it was anticipated in [7] that examination of dipole moments would shed further light on the bound state picture of black holes and p-branes. An extensive study of dipole moments for strings and black holes in an $N=4$ context found complete agreement between the gyromagnetic ratios of states in both short and intermediate multiplets [8]. However, in that work it was realized that the gyromagnetic ratios for short multiplets are completely determined based on supersymmetry alone. Thus, as long as $N=4$ supersymmetry is unbroken, the g-factors must necessarily agree between corresponding supersymmetric black holes and $N_R=1/2$ heterotic string states, and hence do not provide a true test of the strings as black holes conjecture. On the other hand, supersymmetry becomes a lot less restrictive for intermediate and long multiplets. While the correspondence between $N_R>1/2$ states and nonextremal black holes is not so clear, application of T-duality allowed a comparison of g-factors for intermediate black holes and corresponding type-II string states, where agreement was found [8].

This issue of how much freedom is actually present in the gyromagnetic ratios has motivated us to examine both electric and magnetic dipole moment sum rules in a more general extended supersymmetry context. Thus in the following we extend the results of [6] and derive completely general N

*Email: giannak@theory.rockefeller.edu
†Email: jtliu@theory.rockefeller.edu

$=2$ dipole sum rules for particles in either short or long multiplets of $N=2$ supersymmetry. We find the interesting result that, contrary to expectations, the $N=2$ sum rules are weaker than the $N=1$ case in that they depend on additional quantities (certain scalar expectation values) beyond just the mass, electric charge and central charge of the representation. This additional freedom disappears under certain mild assumptions, in which case the $N=2$ sum rules become a simple generalization of the $N=1$ case. In particular, we find that the supersymmetry generated spin adds to the magnetic (electric) dipole moment with a factor of $g=2$ ($g_e=0$).

In the next section we set our notations by discussing the $N=2$ supersymmetry algebra and its representations, while in Sec. III we briefly discuss the linear multiplet of $N=2$ supersymmetry. Finally, in Sec. IV we derive model independent sum rules for the gyromagnetic and gyroelectric ratios of the members of an $N=2$ supermultiplet. These sum rules are presented in both a completely general fashion, and also in simplified form whenever the assumptions mentioned above are valid. Concluding remarks are presented in Sec. V.

II. $N=2$ SUPERSYMMETRY ALGEBRA

Our starting point is the $N=2$ supersymmetry algebra, which admits a single complex central charge $Z=U+iV$. Using the $N=2$ Majorana condition $\overline{Q^i}=i\epsilon^{ij}Q_j^T(C\gamma_5)$, the algebra may be expressed as

$$\{Q_{\alpha i},Q_{\beta j}\}=-2i(\gamma^\mu\gamma_5 C)_{\alpha\beta}\epsilon_{ij}P_\mu + i\epsilon_{ij}(\gamma_5 C)_{\alpha\beta}U-\epsilon_{ij}C_{\alpha\beta}V, \tag{1}$$

where $i,j=1,2$ are SU(2) indices and C is the charge conjugation matrix obeying $C\gamma^\mu C^{-1}=-\gamma^{\mu T}$ and $C^2=-1$. For a massive single particle state, we may work in the rest frame $P^\mu=(M,0,0,0)$. Defining chiralities

$$\gamma_5 Q_i^L=-Q_i^L, \quad \gamma_5 Q_i^R=Q_i^R, \tag{2}$$

and helicities

$$\gamma^{12}Q_{\pm(1/2)i}=\mp iQ_{\pm(1/2)i}, \tag{3}$$

the supersymmetry algebra can be recast as follows:

$$\{Q^L_{\pm(1/2)1},Q^L_{\mp(1/2)2}\}=\mp iZ, \quad \{Q^R_{\pm(1/2)1},Q^R_{\mp(1/2)2}\}=\pm i\overline{Z},$$

$$\{Q^R_{\pm(1/2)1},Q^L_{\mp(1/2)2}\}=-2iM, \quad \{Q^L_{\pm(1/2)1},Q^R_{\mp(1/2)2}\}=2iM, \tag{4}$$

indicating the expected splitting between mass and central charge terms in a Weyl basis.[1]

The above $N=2$ algebra may be diagonalized by introducing the linear combinations

$$Q^{\pm}_{(1/2)i}=\frac{1}{\sqrt{2}}[Q^L_{(1/2)i}\mp ie^{i\alpha}Q^R_{(1/2)i}],$$

$$Q^{\pm}_{-(1/2)i}=\frac{1}{\sqrt{2}}[\pm iQ^R_{-(1/2)i}-e^{-i\alpha}Q^L_{-(1/2)i}], \tag{5}$$

where α is the phase of the central charge, $Z=e^{i\alpha}|Z|$. In terms of these mixed chirality supercharges, the algebra (4) now takes the simple form

$$\{Q^+_{\pm(1/2)1},Q^+_{\mp(1/2)2}\}=2M+|Z|,$$

$$\{Q^-_{\pm(1/2)1},Q^-_{\mp(1/2)2}\}=2M-|Z|, \tag{6}$$

indicating explicitly the $N=2$ Bogomol'nyi bound, $2M\geqslant|Z|$. Massive representations thus split up into either long or short multiplets, with the latter corresponding to saturation of the Bogomol'nyi bound, $2M=|Z|$.

For a long representation, we may rescale the supercharges according to $q^{\pm}_{\pm(1/2)i}=(2M\pm|Z|)^{-1/2}Q^{\pm}_{\pm(1/2)i}$ to recover the Clifford algebra for four fermionic degrees of freedom. From the form of this algebra it follows that one can construct its irreducible representations by starting with a superspin j Clifford vacuum, $|j\rangle$, annihilated by $q^{\pm}_{\pm(1/2)2}$, and acting on it with the creation operators $q^{\pm}_{\pm(1/2)1}$. As a result, we see that the long representation has dimension $(2j+1)\times 2^4$ where $2j+1$ is the degeneracy of the original spin j state. The spins of the states are given by the addition of angular momenta, $j\times[(1)+4(1/2)+5(0)]$, giving generically states of spins $j-1$ to $j+1$ with degeneracies 1, 4, 5+1, 4, 1 (provided $j\geqslant 1$).

When the Bogomol'nyi bound is saturated, $2M=|Z|$, the supercharges $Q^-_{\pm(1/2)i}$ are represented trivially and the algebra becomes the algebra of two fermionic annihilation and creation operators. The short representations thus contain spins $j\times[(1/2)+2(0)]$ (generically giving spins $j-1/2$ to $j+1/2$ with degeneracies 1,2,1) and have dimension $(2j+1)\times 2^2$. These short multiplets of $N=2$ supersymmetry correspond to the same particle content as massive supermultiplets of $N=1$, and in fact satisfy identical dipole moment sum rules, as will be demonstrated below.

Since the supercharges $Q^{\pm}_{(1/2)1}$, $Q^{\pm}_{-(1/2)1}$ are operators of spin 1/2, this leads to a natural shorthand notation for labeling the states of a generic long $N=2$ multiplet. We denote the superspin j Clifford vacuum by $|00\rangle$ where the first (second) entry corresponds to the action of the $2M+|Z|$ ($2M-|Z|$) normalized creation or annihilation algebra of Eq. (6). Acting on this state with the normalized supercharges $q^+_{(1/2)1}$ or $q^+_{-(1/2)1}$ then results in the spin "up" or "down" states $|\uparrow 0\rangle$ or $|\downarrow 0\rangle$, respectively. On the other hand, the action of $q^-_{(1/2)1}$ or $q^-_{-(1/2)1}$ results in the states $|0\uparrow\rangle$ or $|0\downarrow\rangle$. The action of several q's on the Clifford vacuum are then repre-

[1]To fix our phase conventions, we work in the Dirac representation for the γ-matrices and take $\gamma_5=i\gamma^0\gamma^1\gamma^2\gamma^3$ and $C=i\gamma^0\gamma^2$. The spinors then decompose as $\sqrt{2}Q^T_{\alpha i}=Q^L_{(1/2)i}[1\ 0\ 1\ 0]+Q^L_{-(1/2)i}[0\ 1\ 0\ 1]+Q^R_{(1/2)i}[-1\ 0\ 1\ 0]+Q^R_{-(1/2)i}[0\ 1\ 0\ -1]$.

sented in a similar manner. For example the action of all four supercharges is denoted by $|\updownarrow\updownarrow\rangle$. Note that states in a short multiplet will always have a 0 in the second entry. Finally, it should be kept in mind that the physical states of the supermultiplet correspond to the addition of angular momentum j to the above spin states using the appropriate Clebsch-Gordan coefficients.

III. CONSERVED CURRENTS IN SUPERSYMMETRIC THEORIES

For $N=1$ supersymmetry, any conserved current commuting with the supersymmetry generators must belong to a real linear multiplet. In the present case this generalizes to a $N=2$ linear multiplet [9] consisting of $(K^\alpha,\xi_i,S,P,J_\mu)$ where K^α is a SU(2) triplet scalar, ξ_i is a SU(2) doublet Majorana spinor and S,P are real scalars. We take the linear multiplet to transform without central charge, so that the current is conserved, $\partial^\mu J_\mu=0$. As a result the multiplet includes 8 bosonic and 8 fermionic degrees of freedom. The transformation properties of the components under a supersymmetry variation are given by

$$\delta K^\alpha=-\overline{\epsilon^i}\sigma^\alpha_{ij}\xi_j\,,$$

$$\delta\overline{\xi_i}=-\overline{\epsilon^i}(S+\gamma_5P-\gamma^\mu J_\mu)+\overline{\epsilon^j}\sigma^\alpha_{ji}\gamma^\mu\partial_\mu K^\alpha,$$

$$\delta S=-\overline{\epsilon^i}\gamma^\mu\partial_\mu\xi_i\,,\quad \delta P=\overline{\epsilon^i}\gamma^\mu\gamma_5\partial_\mu\xi_i\,,$$

$$\delta J_\mu=-\overline{\epsilon^i}\gamma_{\mu\nu}\partial^\nu\xi_i\,. \tag{7}$$

It follows that two successive supersymmetry transformations on the conserved current J_μ give

$$\delta_\eta\delta_\epsilon J_\mu=i\overline{\epsilon^i}\gamma_{\mu\nu}\partial^\nu[(S+\gamma_5P-\gamma^\lambda J_\lambda)\delta^j_i + i\gamma^\lambda\partial_\lambda K^\alpha\sigma^\alpha_{ij}]\eta_j\,. \tag{8}$$

The matrix elements of this equation between states which belong to the same $N=2$ multiplet give rise to sum rules for the gyromagnetic and gyroelectric ratios of the particle states.

In order to derive both electric and magnetic dipole sum rules, we need the following expansions for the matrix elements of J_μ:

$$\langle j',m',\vec p|J_0|j,m,0\rangle=2Me_j\delta_{jj'}\delta_{mm'} +2Mp_i\langle j',m',0|d^i|j,m,0\rangle+O(p^2),$$

$$\langle j',m',\vec p|J_i|j,m,0\rangle=-e_jp_i\delta_{jj'}\delta_{mm'}-2iM\epsilon_{ijk}p_j \times\langle j',m',0|\mu^k|j,m,0\rangle+O(p^2). \tag{9}$$

When J_μ is the electromagnetic current, $e,\vec d,\vec\mu$ are the electric charge, the electric dipole moment, and the magnetic dipole moment, respectively. Our notation follows [6], where $|j,m,\vec p\rangle$ corresponds to a single particle state of spin j, z-component of spin m and 3-momentum $\vec p$. We emphasize that the expansion of the matrix elements of the current in powers of the momenta is based solely on current conservation. As a convenience, whenever j and m are not explicitly needed, we use the shorthand notation $|j,m,\vec p\rangle=|\alpha,\vec p\rangle$.

IV. DERIVATION OF THE $N=2$ SUM RULES

In [6], the $N=1$ magnetic moment sum rule was derived by noting that a generic double supersymmetry variation may be expressed as

$$\delta_\eta\delta_\epsilon\hat{\mathcal O}=[\overline{\eta}Q,[\overline{\epsilon}Q,\hat{\mathcal O}]] =\overline{\eta}Q\overline{\epsilon}Q\hat{\mathcal O}-\overline{\eta}Q\hat{\mathcal O}\overline{\epsilon}Q-\overline{\epsilon}Q\hat{\mathcal O}\overline{\eta}Q+\hat{\mathcal O}\overline{\epsilon}Q\overline{\eta}Q. \tag{10}$$

Evaluating this expression between given single particle states $\langle\alpha|$ and $|\beta\rangle$, and noting that the supercharge Q generates superpartners $(Q|\alpha\rangle\sim|\tilde\alpha\rangle)$, we are then able to relate matrix elements of $\hat{\mathcal O}$ between different states of a supermultiplet, provided $\delta_\eta\delta_\epsilon\hat{\mathcal O}$ is known. The magnetic dipole sum rules then follow by choosing $\hat{\mathcal O}$ to be the conserved current J_μ, and using Eq. (9) to determine its matrix elements.

This general procedure is simplified in practice by choosing the global supersymmetry transformation parameters η and ϵ in such a way so that several terms on the right hand side of Eq. (10) act as annihilation operators on the initial or final states and hence may be dropped. In particular, to lowest order in momentum $\vec p$, and making use of the Lorentz boost operator $|\alpha,\vec p\rangle=L(\vec p)|\alpha,0\rangle$, we find

$$\begin{aligned}\langle\alpha,\vec p|\delta_\eta\delta_\epsilon J_\mu|\beta,0\rangle=&\langle\alpha,\vec p|J_\mu\overline{\epsilon}Q\overline{\eta}Q|\beta,0\rangle\\ &-\langle\alpha,0|\overline{\epsilon}QL^{-1}(\vec p)J_\mu\overline{\eta}Q|\beta,0\rangle\\ &-\delta_{\mu0}\frac{p^i}{2M}\langle\alpha,0|\overline{\epsilon}\gamma^{0i}QJ_0\overline{\eta}Q|\beta,0\rangle\\ &+O(p^2),\end{aligned} \tag{11}$$

provided $\langle\alpha,\vec p|\overline{\eta}Q=0$. Note that we have used the fact that Q transforms as a spinor so that $[L^{-1}(\vec p),\overline{\eta}Q]=(1/2M)p^i\overline{\eta}\gamma^{0i}Q+O(p^2)$. This expansion of the Lorentz boost gives rise to the last term above, which only contributes to J_0 matrix elements (and hence is only important in deriving the electric dipole moment sum rules).

Turning to the left hand side of Eq. (11), and using the double supersymmetry variation (8), we find

$$\delta_\eta\delta_\epsilon J_i=-i\epsilon_{ijk}p^j\overline{\epsilon} \times[\gamma_\kappa\gamma^5J_0+\gamma_\kappa\gamma^0\gamma^5S+i\gamma_\kappa\gamma^0P]\eta+O(p^2),$$

$$\delta_\eta\delta_\epsilon J_0=-p_i\overline{\epsilon}[\gamma^iJ_0-\gamma^i\gamma_0S-i\gamma^i\gamma_0\gamma^5P]\eta+O(p^2). \tag{12}$$

TABLE I. Matrix elements of μ^3 on the integer spin states of a long multiplet.

$\langle 00\vert$	μ_0							
$\langle\uparrow\uparrow\vert$		$\mu_0+\alpha^++\alpha^-$						
$\langle\downarrow\downarrow\vert$			$\mu_0-\alpha^+-\alpha^-$					
$\langle\uparrow\downarrow\vert$				$\mu_0+\alpha^+-\alpha^-$		$-i\tilde{w}$	$-i\tilde{w}$	
$\langle\downarrow\uparrow\vert$					$\mu_0-\alpha^++\alpha^-$	$-i\tilde{w}$	$-i\tilde{w}$	
$\langle\updownarrow 0\vert$				$i\tilde{w}$	$i\tilde{w}$	μ_0		
$\langle 0\updownarrow\vert$				$i\tilde{w}$	$i\tilde{w}$		μ_0	
$\langle\updownarrow\updownarrow\vert$								μ_0

Note that, while the auxiliary field K^α is unimportant at this order, matrix elements of S and P remain and cannot be ignored. Since the conserved current multiplet commutes with supersymmetry, these matrix elements, like the electric charge, are identical for all states in a given representation. Thus we may define S and P expectations as

$$\langle\alpha,0|S|\beta,0\rangle=2M\mathcal{S}\delta_{\alpha\beta},$$
$$\langle\alpha,0|P|\beta,0\rangle=2M\mathcal{P}\delta_{\alpha\beta}. \tag{13}$$

Since a generic long multiplet of $N=2$ contains many more states than that of $N=1$, we find it convenient to take a systematic approach to examining the matrix elements of Eq. (11) on various states. In particular, the magnetic dipole moment sum rules may be derived in two parts: (i) a set of ''vanishing sum rules'' concerning elements of the dipole moment operator between different states of the multiplet, and (ii) ''diagonal sum rules'' relating diagonal elements of different states.

Derivation of the vanishing sum rules follows by choosing the parameters of transformation to satisfy $\langle\alpha,\vec{p}|\overline{\eta}Q=\langle\alpha,\vec{p}|\overline{\epsilon}Q=0$, in which case Eq. (11) becomes

$$\langle\alpha,\vec{p}|J_i\overline{\epsilon}Q\,\overline{\eta}Q|\beta,0\rangle=\langle\alpha,\vec{p}|\delta_\eta\delta_\epsilon J_i|\beta,0\rangle+O(p^2). \tag{14}$$

By choosing both $\langle\alpha,\vec{p}|$ and $|\beta,0\rangle$ to denote different states of the supermultiplet, this allows us to compute the off-diagonal matrix elements of $\vec{\mu}$ in terms of the charges e, $\mathcal{S}$ and $\mathcal{P}$ that show up in the double variation on the right hand side.

The diagonal sum rules are derived instead by taking states satisfying $\langle\alpha,\vec{p}|\overline{\eta}Q=\overline{\epsilon}Q|\beta,0\rangle=0$. For this case, we find

$$\langle\alpha,0|\overline{\epsilon}QL^{-1}(\vec{p})J_i\overline{\eta}Q|\beta,0\rangle=\langle\alpha,\vec{p}|J_i[\overline{\epsilon}Q,\overline{\eta}Q]|\beta,0\rangle-\langle\alpha,\vec{p}|\delta_\eta\delta_\epsilon J_i|\beta,0\rangle+O(p^2). \tag{15}$$

Note that $[\overline{\epsilon}Q,\overline{\eta}Q]$ corresponds to the supersymmetry algebra, and hence gives $2M\pm|Z|$ for appropriate parameters. Thus, when generating properly normalized superpartners, the above expression simply states that the dipole moment of the superpartner (on the left) is the same as the dipole moment of the original state (on the right) with the addition of a supersymmetry generated correction given by $\delta_\eta\delta_\epsilon J_i$.

We recall that a basic long multiplet contains 16 states, divided into $8+8$ based on integer or half-integer spins. Since the magnetic dipole operator (being a vector) does not connect integer and half-integer spins, its matrix elements on these 16 states split up into two 8×8 block diagonal pieces. Applying both vanishing and diagonal sum rules, the matrix elements of the z-component of $\vec{\mu}$, $\langle\alpha,0|\mu^3|\beta,0\rangle$, are given in Tables I (integer spins) and II (half-integer spins). We have taken, by definition, $\mu_0=\langle 00|\mu^3|00\rangle$. The real numbers $\alpha^\pm$ and $\tilde{w}$ are given by $\alpha^+=(e+v)/(2M+|Z|)$, $\alpha^-=(e-v)/(2M-|Z|)$ and $\tilde{w}=w/\sqrt{4M^2-|Z|^2}$, where v and w are the rotated scalar expectation values

$$\begin{pmatrix} v \\ w \end{pmatrix}=\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \mathcal{S} \\ \mathcal{P} \end{pmatrix}. \tag{16}$$

This is the main result of our paper. We note that the matrix elements of the dipole moment operator between the different states of a long $N=2$ supermultiplet are expressed in terms of the mass M, the central charge $|Z|$ and the charges $e,\mathcal{S},\mathcal{P}$.

TABLE II. Matrix elements of μ^3 on the half-integer spin states of a long multiplet.

$\langle\uparrow 0\vert$	$\mu_0+\alpha^+$	$-i\tilde{w}$						
$\langle 0\uparrow\vert$	$i\tilde{w}$	$\mu_0+\alpha^-$						
$\langle\downarrow 0\vert$			$\mu_0-\alpha^+$	$i\tilde{w}$				
$\langle 0\downarrow\vert$			$-i\tilde{w}$	$\mu_0-\alpha^-$				
$\langle\uparrow\updownarrow\vert$					$\mu_0+\alpha^+$	$i\tilde{w}$		
$\langle\updownarrow\uparrow\vert$					$-i\tilde{w}$	$\mu_0+\alpha^-$		
$\langle\downarrow\updownarrow\vert$							$\mu_0-\alpha^+$	$-i\tilde{w}$
$\langle\updownarrow\downarrow\vert$							$i\tilde{w}$	$\mu_0-\alpha^-$

TABLE III. Matrix elements of d^3 on the integer spin states of a long multiplet.

$\langle 00\|$	d_0							
$\langle\uparrow\uparrow\|$		$d_0+iw^+-iw^-$						
$\langle\uparrow\uparrow\|$			$d_0-iw^++iw^-$					
$\langle\uparrow\downarrow\|$				$d_0+iw^++iw^-$		$\tilde{u}$	$\tilde{u}$	
$\langle\downarrow\uparrow\|$					$d_0-iw^+-iw^-$	$\tilde{u}$	$\tilde{u}$	
$\langle\updownarrow 0\|$				$-\tilde{u}$	$-\tilde{u}$	d_0		
$\langle 0\updownarrow\|$				$-\tilde{u}$	$-\tilde{u}$		d_0	
$\langle\updownarrow\updownarrow\|$								d_0

Short multiplets, on the other hand, are expected to behave as massive $N=1$ multiplets. The relation $2M=|Z|$ which holds for short multiplets implies that the supercharges $Q^-_{\pm(1/2)i}$ are represented trivially and as a result all the states with up or down arrows in the second entry disappear. Then, by picking ϵ and η in Eq. (15) to select the $Q^-_{\pm(1/2)i}$ supercharges, we are left with $\langle\alpha,\vec{p}|\delta_\eta\delta_\epsilon J_i|\beta,0\rangle=0+O(p^2)$. Combined with the explicit supersymmetry variation, Eq. (12), this gives rise to relations between e, $\mathcal{S}$, $\mathcal{P}$, M, and Z. More specifically we find that $e=v$ and $w=0$. These two relations can then be written as one:

$$\mathcal{S}+i\mathcal{P}=\frac{eZ}{2M}. \tag{17}$$

The matrix elements of μ^3 on the states of a short multiplet simplify as follows

$$\begin{matrix}\langle 00| \\ \langle\uparrow 0| \\ \langle\downarrow 0| \\ \langle\updownarrow 0|\end{matrix}\begin{bmatrix}\mu_0 & & & \\ & \mu_0+\frac{e}{2M} & & \\ & & \mu_0-\frac{e}{2M} & \\ & & & \mu_0\end{bmatrix}, \tag{18}$$

in agreement with the results of [6].

In a similar manner we can derive sum rules for the electric dipole moments. The results are summarized in Tables III and IV, where $d_0=\langle 00|d^3|00\rangle$ and $w^\pm=w/(2M\pm|Z|)$ and $\tilde{u}=[(e|Z|/2M)-v]/\sqrt{4M^2-|Z|^2}$. Curiously enough, we see that in general the electric dipole moments are nonvanishing, even with $d_0=0$. Demanding that $N=2$ supersymmetry does not generate an electric dipole moment when none was initially present then requires $0=w^+=w^-=\tilde{u}$, so that in fact $v=e|Z|/2M$ and $w=0$, corresponding to the condition (17) that was found for short multiplets.

While in general we have been unable to ascertain whether or not Eq. (17) must continue to hold for long multiplets, it appears that this condition is true in practice for many explicit $N=2$ realizations. In fact, both magnetic and electric dipole moment sum rules greatly simplify whenever Eq. (17) is valid. To see this, note that in this case $\alpha^+=\alpha^-=e/2M$, so that the magnetic dipole matrix elements of Tables I and II become diagonal, with the addition of $e/2M$ units of dipole moment for every 1/2 unit of spin generated by the supersymmetry algebra. The electric dipole matrix elements of Tables III and IV are even simpler; they only contain the original electric dipole moment d_0, with no addition from supersymmetry.

Finally, we present the g-factor sum rules for the physical states of the superspin j multiplet by adding the supersymmetry generated spin to the original spin j using appropriate Clebsch-Gordan combinations.[2] Recalling that the states of the $N=2$ long multiplet are generated by $j\times[(1)+4(1/2)+5(0)]$, we need the Clebsch-Gordan coefficients for $j\times 1$ and $j\times 1/2$. For example, for the latter, we use

$$\begin{aligned}|j+\tfrac{1}{2},m+\tfrac{1}{2}\rangle&=\frac{1}{\sqrt{2j+1}}\left[\sqrt{j+m+1}|j,m;\tfrac{1}{2},\tfrac{1}{2}\rangle\right.\\&\quad\left.+\sqrt{j-m}|j,m+1;\tfrac{1}{2},-\tfrac{1}{2}\rangle\right],\\|j-\tfrac{1}{2},m+\tfrac{1}{2}\rangle&=\frac{1}{\sqrt{2j+1}}\left[-\sqrt{j-m}|j,m;\tfrac{1}{2},\tfrac{1}{2}\rangle\right.\\&\quad\left.+\sqrt{j+m+1}|j,m+1;\tfrac{1}{2},-\tfrac{1}{2}\rangle\right].\end{aligned} \tag{19}$$

The g-factors may then be defined in terms of the matrix elements of the magnetic dipole moment operator $\vec{\mu}$ between states of definite angular momentum using the Wigner-Eckart theorem as follows:

$$\langle j,m|\mu_3|j,m\rangle=\frac{e}{2M}mg_j, \tag{20}$$

where j and m label any state of angular momentum j and z-component m. Combining Eqs. (19) and (20), and using the matrix elements of Table II, then gives for the four $j+1/2$ and four $j-1/2$ states

$$g_{j+1/2}=g_j+\frac{g_s-g_j}{2j+1},\quad g_{j-1/2}=g_j-\frac{g_s-g_j}{2j+1}, \tag{21}$$

[2]While the addition of angular momentum was an integral part of the derivation of the $N=1$ sum rule [6], we find it more convenient to keep the superpartner generation and the Clebsch-Gordan manipulation separate, especially for large multiplets.

TABLE IV. Matrix elements of d^3 on the half-integer spin states of a long multiplet.

$\langle\uparrow 0\vert$	d_0+iw^+	$\tilde{u}$						
$\langle 0\uparrow\vert$	$-\tilde{u}$	d_0-iw^-						
$\langle\downarrow 0\vert$			d_0-iw^+	$-\tilde{u}$				
$\langle 0\downarrow\vert$			$\tilde{u}$	d_0+iw^-				
$\langle\uparrow\updownarrow\vert$					d_0+iw^+	$-\tilde{u}$		
$\langle\updownarrow\uparrow\vert$					$\tilde{u}$	d_0-iw^-		
$\langle\downarrow\updownarrow\vert$							d_0-iw^+	$\tilde{u}$
$\langle\updownarrow\downarrow\vert$							$-\tilde{u}$	d_0+iw^-

where $g_s=2$ is the supersymmetry generated g-factor, corresponding to $\alpha^+=\alpha^-=e/2M$ whenever Eq. (17) holds.[3] Following the same analysis we find for the $j\times 1$ combination

$$g_{j+1}=g_j+\frac{g_s-g_j}{j+1},\quad g_{j-1}=g_j-\frac{g_s-g_j}{j},$$

$$g_{j'}=g_j+\frac{g_s-g_j}{j(j+1)}. \tag{22}$$

Of the $5+1$ states of spin j, five have a g-factor of g_j, while the last has a g-factor of $g_{j'}$. This demonstrates in particular that in extended supersymmetry not all states of the same spin have to have the same gyromagnetic ratio.

Next we turn our attention to the transition magnetic dipole moments. This time, using the Wigner-Eckart theorem to write

$$\langle j-\tfrac{1}{2},m+\tfrac{1}{2}|\mu_3|j+\tfrac{1}{2},m+\tfrac{1}{2}\rangle = \frac{e}{2M}h_j\sqrt{j(j+1)-m(m+1)}, \tag{23}$$

we find the transition elements

$$h_j=\frac{g_j-g_s}{2j+1},\quad h_{j+1/2}=\sqrt{\frac{j}{2j+1}}\,\frac{g_j-g_s}{j+1},$$

$$h_{j-1/2}=\sqrt{\frac{j+1}{2j+1}}\,\frac{g_j-g_s}{j}, \tag{24}$$

where $h_{j\pm 1/2}$ corresponds to the matrix elements of the dipole moment operator between the states with spins j and $j\pm 1$.

Short representations of $N=2$ have g-factors given by $g_{j\pm 1/2}$ in Eq. (21) and a transition moment given by h_j in Eq. (24). Note that this agrees with the sum rule found in [6] as is expected due to the correspondence of $N=2$ short representations with massive $N=1$ representations.

V. CONCLUSIONS

In the above we have derived model independent sum rules for the gyromagnetic and gyroelectric ratios of particles which belong to a single $N=2$ supermultiplet. As demonstrated in Eqs. (21), (22), and (24), the gyromagnetic ratio of any state in a generic long multiplet may be expressed in terms of the quantities g_j and g_s, where $g_s=2$ whenever the natural relation of Eq. (17) holds. Although we have examined Eq. (17) carefully, we have as yet been unable to determine its validity in a model-independent manner. This leads us to believe that there may indeed be models where $\mathcal{S}+i\mathcal{P}$ are free, thus allowing in addition a $N=2$ supersymmetry contribution to the electric dipole moment, as indicated in Tables III and IV. This novel feature is somewhat surprising in that one would usually anticipate the addition of more symmetry in going from $N=1$ to $N=2$ to lead to more restrictions and thus stronger sum rules on the dipole moments. However, this is the opposite of what is actually found above. Furthermore, there is no contradiction with the sum rule determined for the $N=1$ subalgebra of $N=2$. In particular, noting that

$$|\uparrow\rangle_{N=1}=\frac{1}{\sqrt{2}}\left[\sqrt{1+\frac{|Z|}{2M}}\,|\uparrow 0\rangle-\sqrt{1-\frac{|Z|}{2M}}\,|0\uparrow\rangle\right], \tag{25}$$

we find the expected result $\langle\uparrow|\mu^3|\uparrow\rangle_{N=1}=\mu_0+e/2M$ independent of $\mathcal{S}+i\mathcal{P}$.

Just as the short representation of $N=2$ is connected with the massive $N=1$ representation, the short representation of $N=4$ (preserving half of the supersymmetries) is connected with the long representation of $N=2$. Thus we expect the sum rules derived herein to also apply to the short $N=4$ case. Since the latter were studied in [8], we may contrast the two approaches. While the present derivation is quite general, and focuses on a conserved current commuting with supersymmetry, the latter took an explicit $N=4$ (on-shell only) supergravity coupled Yang-Mills theory and studied the dipole moments of BPS states through their asymptotic field behaviors (although black holes were studied in [8], the sum rules were derived in a general fashion, and depend only on having an appropriate supersymmetric field configura-

[3]In the more general case g_s may be determined in terms of the eigenvalues of the magnetic dipole matrix, and would take on two different values, with the four $(j+1/2,j-1/2)$ pairs splitting into two plus two pairs.

tion). The resulting sum rule found in [8] corresponds to the present $N=2$ long case, with $g_j=0$ and $g_s=2$. In particular, this indicates that the g-factors for the $N=4$ short case are completely fixed, hinting at the possibility that stronger sum rules do arise in $N=4$ and $N=8$ theories (where the latter only has graviphotons) that are not yet apparent in the $N=2$ case.

Finally, note that the technique of [8] allows a determination of g-factor sum rules for graviphotons, which is not possible in the present framework (since graviphotons do not commute with supersymmetry). It would be interesting, however, to see if model independent sum rules for graviphoton couplings could be determined by working with a supergravity multiplet instead of a real linear multiplet.

ACKNOWLEDGMENTS

We would like to thank V. P. Nair and M. Porrati for useful discussions. This work was supported in part by the Department of Energy under Contract Number DE-FG02-91ER40651-TASKB.

[1] S. Weinberg, *Lectures on Elementary Particles and Quantum Field Theory* (MIT Press, Cambridge, MA, 1970).
[2] S. Ferrara, M. Porrati, and V. Telegdi, Phys. Rev. D **46**, 3529 (1992).
[3] L. Singh and C. Hagen, Phys. Rev. D **9**, 898 (1974); **9**, 910 (1974); F. Belinfante, Phys. Rev. **92**, 997 (1953); K. Case, *ibid.* **94**, 1442 (1954); C. Hagen and W. Hurley, Phys. Rev. Lett. **24**, 1381 (1970).
[4] S. Ferrara and E. Remiddi, Phys. Lett. **53B**, 347 (1974).
[5] C. L. Bilchak, R. Gastmans, and A. Van Proeyen, Nucl. Phys. **B273**, 46 (1986).
[6] S. Ferrara and M. Porrati, Phys. Lett. B **288**, 85 (1992).
[7] M. J. Duff and J. Rahmfeld, Nucl. Phys. **B481**, 332 (1996).
[8] M. J. Duff, J. T. Liu, and J. Rahmfeld, Nucl. Phys. **B494**, 161 (1997).
[9] M. Sohnius, Phys. Lett. **81B**, 8 (1979); D. Breitenlohner and M. Sohnius, Nucl. Phys. **B165**, 483 (1980).